$\mathcal{B}(H)$ has a pure state that is not multiplicative on any masa

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Assuming the continuum hypothesis, we prove that $\mathcal{B}(H)$ has a pure state whose restriction to any masa is not pure. This resolves negatively old conjectures of Kadison and Singer and of Anderson.

bounded operators | Hilbert space

Let H be a separable infinite-dimensional Hilbert space and let $\mathcal{B}(H)$ be the algebra of bounded operators on H. Kadison and Singer (1) suggested that every pure state on $\mathcal{B}(H)$ would restrict to a pure state on some maximal abelian self- adjoint subalgebra (masa). Anderson (2) formulated the stronger conjecture that every pure state on $\mathcal{B}(H)$ is *diagonalizable*, that is, of the form $f(A) = \lim_{\mathcal{U}} \langle Ae_n, e_n\rangle$ for some orthonormal basis (e_n) and some ultrafilter $\mathcal{U}$ over $\mathbf{N}$.

An *atomic masa* is the set of all operators that are diagonalized with respect to some given orthonormal basis of H. Anderson's conjecture is related to a fundamental problem in C*-algebras also raised in ref. 1 and now known as the Kadison–Singer problem, which asks whether every pure state on an atomic masa of $\mathcal{B}(H)$ has a unique extension to a pure state on $\mathcal{B}(H)$. If (e_n) is an orthonormal basis of H, then every pure state f_0 on the corresponding atomic masa $\mathcal{M}$ has the form $f_0(A) = \lim_{\mathcal{U}}\langle Ae_n, e_n\rangle$ for some ultrafilter $\mathcal{U}$ over $\mathbf{N}$ and all $A \in \mathcal{M}$, and Anderson (3) showed that the same formula, now for $A \in \mathcal{B}(H)$, defines a pure state f on $\mathcal{B}(H)$. Thus, a positive solution to the Kadison–Singer problem would say that f is the only pure state on $\mathcal{B}(H)$ that extends f_0.

In the presence of a positive solution to the Kadison–Singer problem, Anderson's conjecture is equivalent to the weaker statement that every pure state on $\mathcal{B}(H)$ restricts to a pure state on some atomic masa. However, assuming the continuum hypothesis, we show that this weaker statement is false; in fact, there exist pure states on $\mathcal{B}(H)$ whose restriction to any masa is not pure. It follows that there are pure states on $\mathcal{B}(H)$ that are not diagonalizable. It seems likely that the statement "every pure state on $\mathcal{B}(H)$ restricts to a pure state on some atomic masa" is also consistent with standard set theory. This, together with a positive solution to the Kadison–Singer problem, would imply the consistency of a positive answer to Anderson's conjecture.

The key lemma we need is the following. Let $\mathcal{K}(H)$ be the algebra of compact operators on H, let $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ be the Calkin algebra, and let $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ be the natural quotient map. We also write $\dot{a}$ for $\pi(a)$, for any $a \in \mathcal{B}(H)$.

Lemma 0.1. *Let $\mathcal{A}$ be a separable C*-subalgebra of $\mathcal{B}(H)$ which contains $\mathcal{K}(H)$, let f be a pure state on $\mathcal{A}$ that annihilates $\mathcal{K}(H)$, and let $\mathcal{M}$ be a masa of $\mathcal{B}(H)$. Then there is a pure state g on $\mathcal{B}(H)$ that extends f and whose restriction to $\mathcal{M}$ is not pure.*

Proof: By Proposition 6 of ref. 4 we can find an infinite-rank projection $p \in \mathcal{B}(H)$ such that

$$\dot{p}\dot{a}\dot{p} = f(a)\dot{p} \qquad \textbf{[1]}$$

for all $a \in \mathcal{A}$.

Lemma 1.4 and Theorem 2.1 of ref. 5 imply that $\pi(\mathcal{M})$ is a masa of $\mathcal{C}(H)$. It follows that there is a projection $q \in \mathcal{M}$ such that $\dot{q}$ neither contains nor is orthogonal to $\dot{p}$. Otherwise $\dot{p}$ would be in the commutant of $\pi(\mathcal{M})$, and hence would belong to $\pi(\mathcal{M})$ by maximality. But this would mean $\dot{p}$ is minimal in $\pi(\mathcal{M})$ because any nonzero projection below $\dot{p}$ neither contains nor is orthogonal to $\dot{p}$, and $\pi(\mathcal{M})$ has no minimal projections.

Let $\phi : \mathcal{C}(H) \to \mathcal{B}(K)$ be an irreducible representation of the Calkin algebra. It is faithful because $\mathcal{C}(H)$ is simple. Therefore, $\phi(\dot{q})$ neither contains nor is orthogonal to $\phi(\dot{p})$, so we can find a unit vector $v \in K$ in the range of $\phi(\dot{p})$ which is neither contained in nor orthogonal to the range of $\phi(\dot{q})$. Finally, define $g(a) = \langle\phi(\dot{a})v, v\rangle$ for all $a \in \mathcal{B}(H)$. This is a pure state on $\mathcal{B}(H)$ because $\phi \circ \pi$ is an irreducible representation of $\mathcal{B}(H)$. It extends f because, using Eq. **1**,

$$\begin{aligned} g(a) = \langle\phi(\dot{a})v, v\rangle = \langle\phi(\dot{a})\phi(\dot{p})v, \phi(\dot{p})v\rangle = \langle\phi(\dot{p}\dot{a}\dot{p})v, v\rangle \\ = \langle f(a)\phi(\dot{p})v, v\rangle = f(a) \end{aligned}$$

for all $a \in \mathcal{A}$. Finally, its restriction to $\mathcal{M}$ is not pure because the projection $q \in \mathcal{M}$ has the property that

$$g(q) = \langle\phi(\dot{q})v, v\rangle$$

is strictly between 0 and 1, since v is neither contained in nor orthogonal to the range of $\phi(\dot{q})$. □

Theorem 0.2. *Assume the continuum hypothesis. Then there is a pure state on $\mathcal{B}(H)$ whose restriction to any masa is not pure.*

Proof: Let (a_α), $\alpha < \aleph_1$, enumerate the elements of $\mathcal{B}(H)$. Since every von Neumann subalgebra of $\mathcal{B}(H)$ is countably generated, a simple cardinality argument shows that there are only $\aleph_1$ such subalgebras. Hence, $\mathcal{B}(H)$ has only $\aleph_1$ masas. Let $(\mathcal{M}_\alpha)$, $\alpha < \aleph_1$, enumerate the masas of $\mathcal{B}(H)$.

We now inductively construct a nested transfinite sequence of unital separable C*-subalgebras $\mathcal{A}_\alpha$ of $\mathcal{B}(H)$ together with pure states f_α on $\mathcal{A}_\alpha$ such that for all $\alpha < \aleph_1$

1. $a_\alpha \in \mathcal{A}_{\alpha+1}$.
2. if $\beta < \alpha$ then f_α restricted to $\mathcal{A}_\beta$ equals f_β.
3. $\mathcal{A}_{\alpha+1}$ contains a projection $q_\alpha \in \mathcal{M}_\alpha$ such that $0 < f_{\alpha+1}(q_\alpha) < 1$.

Begin by letting $\mathcal{A}_0$ be any separable C*-subalgebra of $\mathcal{B}(H)$ that is unital and contains $\mathcal{K}(H)$ and let f_0 be any pure state on $\mathcal{A}_0$ that annihilates $\mathcal{K}(H)$. At successor stages, use the lemma to find a projection $q_\alpha \in \mathcal{M}_\alpha$ and a pure state g on $\mathcal{B}(H)$ such that $g|\mathcal{A}_\alpha = f_\alpha$ and $0 < g(q_\alpha) < 1$. By Lemma 4 of ref. 6 there is a separable C*-algebra $\mathcal{A}_{\alpha+1} \subseteq \mathcal{B}(H)$ that contains $\mathcal{A}_\alpha$, a_α, and q_α, and such that the restriction $f_{\alpha+1}$ of g to $\mathcal{A}_{\alpha+1}$ is pure. To see this, write $\mathcal{B}(H)$ as the union of a continuous nested transfinite sequence of separable C*-algebras $\mathcal{B}_\gamma$ such that $\mathcal{B}_0$ is the C*-algebra

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generated by $\mathcal{A}_\alpha$, a_α, and q_α. The cited lemma guarantees that the restriction of g to some $\mathcal{B}_\gamma$ will be pure. Thus, the construction may proceed. At limit ordinals α, let $\mathcal{A}_\alpha$ be the closure of $\cup_{\beta<\alpha} \mathcal{A}_\beta$. The state f_α is determined by the condition $f_\alpha|\mathcal{A}_\beta = f_\beta$, and it is easy to see that f_α must be pure. [If g_1 and g_2 are states on $\mathcal{A}_\alpha$ such that $f_\alpha = (g_1 + g_2)/2$, then for all $\beta < \alpha$ purity of f_β implies that g_1 and g_2 agree when restricted to $\mathcal{A}_\beta$; thus $g_1 = g_2$.] This completes the description of the construction.

Now define a state f on $\mathcal{B}(H)$ by letting $f|_{\mathcal{A}_\alpha} = f_\alpha$. By the reasoning used immediately above, f is pure, and since $0 < f(q_\alpha) < 1$ for all α, the restriction of f to any masa is not pure.

It is interesting to contrast Theorem 0.2 with Theorem 9 of ref. 4, which states that (assuming the continuum hypothesis) any state on $\mathcal{C}(H)$ restricts to a pure state on some masa of $\mathcal{C}(H)$. This does not conflict with our result because there are many masas of $\mathcal{C}(H)$ that do not come from masas of $\mathcal{B}(H)$ (regardless of the truth of the continuum hypothesis). Indeed, $\mathcal{B}(H)$ has $2^{\aleph_0}$ masas but $\mathcal{C}(H)$ has $2^{2^{\aleph_0}}$ masas. This can be seen by first finding $2^{\aleph_0}$ mutually orthogonal nonzero projections p_α in $\mathcal{C}(H)$, then finding projections $q_\alpha^1, q_\alpha^2 < p_\alpha$ such that $q_\alpha^1 q_\alpha^2 \neq q_\alpha^2 q_\alpha^1$ for each α, and finally for each set $S \subseteq 2^{\aleph_0}$ choosing a masa of $\mathcal{C}(H)$ that contains $\{q_\alpha^1 : \alpha \in S\}$ and $\{q_\alpha^2 : \alpha \subseteq S\}$. It is easy to see that this produces $2^{2^{\aleph_0}}$ distinct masas.

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